

Lattice Conformal Theories and Their Integrable Perturbations

Alexander V. Antonov^{* †}

*Landau Institute for Theoretical Physics,
Kosygina 2, GSP-1, 117940 Moscow V-334, Russia*

Alexander A. Belov[‡]

*International Institute for Earthquake Prediction Theory
and Mathematical Geophysics,
Warshavskoe sh., 79, k. 2, Moscow 113556, Russia*

Karen D. Chaltikian[§]

*Department of Physics,
Stanford University,
Stanford, CA 94305-4060, USA*

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Abstract

We consider lattice analogues of some conformal theories, including WZW and Toda models. We describe discrete versions of Drinfeld-Sokolov reduction and Sugawara construction for the WZW model. We formulate perturbation theory in chiral sector. We describe the Spaces of Integrals of Motion in the perturbed theories. We interpret the perturbed WZW model in terms of NLS-hierarchy and obtain an embedding of this model into the lattice KP-hierarchy.

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[†]E-mail: antonov@landau.ac.ru

[‡]Supported in part by International Science Foundation (Grant N89000).

[§]E-mail: karen@quantum.stanford.edu

This work turned out to be the last one in which Sasha Belov took immediate part. To our deep grief, he passed away at only 32 on March, 7, 1995. Up to his last moments he continued working and thinking about the problems going far beyond the scope of this paper.

His great interest in developing this new direction and bright personality made any communication with him extremely interesting and enriching. Always tuned to learning the latest achievements in any direction, he developed his own unique style of thinking and approaching any problem. Unfortunately, most of his brilliant ideas and exciting results still remain in his notebooks. We hope most of them will be published sooner or later.

It is no doubt that continuing to work on lattice analogues of Conformal Theories and Integrable Models we still will be inspired by his ideas for a very long time.

A. Antonov, K. Chaltikian

1 Introduction

In a recent series of papers by Feigin and Frenkel [1, 2, 3] a new approach to studying integrable systems was developed. In particular, a new procedure of construction of quantum integrals of motion was proposed, based on some nice homological constructions. It provides the universal treatment of conserved quantities in both conformal theories and massive integrable models, whenever some extended symmetries are present in the theory. In many interesting situations in quantum regime this extended symmetry is usually described by some Quantum Group (see [16, 14] for a review) which also appear independently in Quantum Inverse Scattering Method [13] of study of integrable models. The relationship between different quantum group symmetries in one and the same model or in different but connected by sort of limiting transition models is of great interest nowadays. Present paper applies Feigin-Frenkel method to the lattice analogues of some conformal theories, including WZW - model and their perturbations as well as associated integrable hierarchies.

There are at least three reasons for special interest in lattice WZW model. They are mainly the same that dictate lattice reformulation of any quantum model. But WZW model has some peculiar features which make these three aspects deeply interrelated. Let us briefly sketch these reasons for lattice setting of WZW - model.

A. Regularization. Rigorous quantum consideration of WZW model (as well as pure σ -model) involves renormalization for the ultraviolet behavior of the naively formulated model is ill-defined. The lattice regularization is the most natural one in a sense that it preserves the symmetries of continuous model (or , better to say, substitutes these symmetries by appropriate lattice analogues). Among these symmetries are the gauge and conformal invariance of the model. The other useful feature of the lattice regularization is that it may be introduced already on the classical level. Thus "quantization" results in "q-deformation" of the model and Poisson-Lie symmetry of classical lattice appears to be "quasiclassical limit" of quantum group symmetry. It should be mentioned that originally lattice conformal symmetry has been introduced in the framework of analogous regularization for quantum Liouville and Toda theories [17, 21, 19, 20]. The main contribution to formulation of the lattice WZW - model has been done by St.-Petersburg group in the papers [26, 27, 28]. Further development of the formalism including the first free field representation has been made by Falceto and Gawedzki [29]. The introduction of the lattice Kac-Moody algebra (LKMA) and associated quantum lattice monodromy [26, 27, 28] brings us to the next aspect of lattice formulation, which is

B. Unification of symmetries, or more precisely, coupling of the quantum group symmetries to the space-time degrees of freedom. Probably, LKMA as it appears in lattice WZW model is the best way of unification of quantum-group and space-time symmetries of the model. According to ideology of modern string theory all the symmetries of the model should be considered on equal footing.

C. Calculability. The third essential reason for lattice setting of any field theoretical model is calculability of the otherwise ill-defined certain physical quantities. This is the main reason for discretization of Yang-Mills model as well as low-dimensional quantum gravity. It is well-known that there is a deep connection between (continuous) WZW model and topological Chern-Simons theory [23, 24]. Certain expectation values of continuum Chern-Simons theory may be computed as lattice statistical sums defined in terms of quantum groups.

As we have seen all the three aspects of setting of quantum theory on the lattice (regular-

ization, unification of symmetries and calculability) are present in WZW case. Their mutual relation originates from the fact that all of them are intrinsically connected with quantum groups.

The aim of this work is twofold: to push forward the understanding of lattice conformal theories, especially of lattice WZW model and to study analogues of their integrable perturbations.

The paper is organized as follows. In Sec.2 on a simple example we review the main ideas of Feigin-Frenkel approach to the description of integrable systems. In Sec. 3 we remind the St.-Petersburg definition of lattice KM algebra, introduce convenient analogue of the Chevalley basis and describe the free fields representation of the lattice WZW model. In Sec.4 we describe explicitly lattice Drinfeld-Sokolov reduction and in Sec.5 – lattice Sugawara construction. Then in Sec.6 we proceed with perturbation theory for lattice WZW model. For the sake of simplicity we restrict ourselves with \widehat{sl}_2 case. All our considerations are undertaken on quasiclassical level. We describe lattice Maxwell-Bloch (MB) system. We also propose to look at this system as at the proper lattice analogue of NLS hierarchy. Sec.7 is devoted to study of connection between lattice NLS and “universal” lattice KP hierarchies. We find that lattice NLS hierarchy may be understood as a special two-field realization of the latter through a certain embedding. We also discuss lattice affine Toda theories and describe their spaces of conservation laws. We end up with some concluding remarks and reviewing of unsolved questions.

2 Feigin-Frenkel approach

In this section we briefly review the main ideas of the cohomological approach following the papers [1, 2, 3, 18].

Let $\pi = \oplus_i \pi_i$ be a graded phase space with Poisson structure $\{, \}_\pi : \pi \otimes \pi \rightarrow \pi$. The space of the local functionals,

$$\mathcal{F} = \oplus_i \mathcal{F}_i$$

is related to π by the integral mapping:

$$\int : \pi_k \longrightarrow \mathcal{F}_k$$

Suppose there is given an action of nilpotent part $n_+(\mathcal{G})$ of some semisimple (affine) algebra \mathcal{G} (resp. $\widehat{\mathcal{G}}$) on π_0 . The kernel of such an action of $n_+(\mathcal{G})$ as will be referred to as the *Space of local integrals of motion* (IM). Traditionally, the generators of the action of $n_+(\mathcal{G})$ are called screening charges (SC). Thus, the space of local IM's is given by an intersection of kernels of all SC's of the model. In finite-dimensional case it is possible to find a set of local fields, spanning the space

$$\ker_{\pi_0}(n_+(\mathcal{G})). \quad (2.1)$$

Depending on the particular free field realization of SC's, one can obtain the corresponding W -algebra [7, 6] or Kac-Moody algebra [33, 34, 35, 36]. In the infinite-dimensional case, when SC's generate affine algebra $\widehat{\mathcal{G}}$, we have the space of local IM

$$IM = \ker_{\mathcal{F}_0}(n_+(\widehat{\mathcal{G}})). \quad (2.2)$$

The most well-known examples are coming from conformal field theory ([4, 5, 8, 7, 9, 36]). and theory of non-linear integrable equations (see [11] and references therein). In the semisimple case, with phase space being the vertex operator algebra corresponding to some Cartan subalgebra we get nothing but the Gelfand-Dickey algebra (or its quantum deformation, $W(\mathcal{G})$ -algebra, in quantum theory). This algebra can be viewed as a zeroth cohomology of a certain complex, resembling the Bernstein-Gelfand-Gelfand resolution of the trivial representation of \mathcal{G} . Following the same procedure in the affine case one obtains a set of conservation laws in involution with respect to Gelfand-Dickey Poisson structure (or with respect to quantum commutation relations [7, 6, 10]).

As a simple example, consider the case $sl(2)$ in classical limit. In free field representation the phase space components π_k are defined as

$$\pi_k := \frac{\text{Differential}}{\text{Polynomials}} \left(i\partial\varphi(x) \otimes e^{ik\varphi} \right)$$

where $\varphi(x)$ is a free bosonic field with Poisson bracket

$$\{\varphi(x), \varphi(y)\} = \text{sign}(x - y)$$

One SC

$$Q \equiv \int dx e^{i\varphi(x)} \quad \pi_0 \xrightarrow{Q} \pi_1$$

makes nilpotent subalgebra of semisimple $sl(2)$. Together with the second one

$$\begin{array}{ccc} & \pi_1 & \\ & \nearrow Q & \\ \pi_0 & & \\ & \searrow \overline{Q} & \\ & \pi_{-1} & \end{array}$$

they form nilpotent subalgebra of \widehat{sl}_2 – one actually has to check that Q and \overline{Q} satisfy the Serre's relations

$$ad_Q^3(\overline{Q}) = 0 = ad_{\overline{Q}}^3(Q)$$

Action of the SC's is defined as the adjoint action with respect to the Poisson bracket (or with respect to the commutator in quantum case). It is easy to check, that $\ker_{\pi_0}(Q)$ is spanned by the element

$$u = \frac{1}{2}(i\partial\varphi(x))^2 + i\partial^2\varphi,$$

which satisfies the second Gel'fand-Dickey structure for the KdV equation (or Virasoro algebra in quantum case)

$$\{u(x), u(y)\} = \left(u(x)\partial_x + \partial_x u(x) - \partial_x^3 \right) \delta(x - y)$$

$\ker_{\mathcal{F}_0}(Q) \cap \ker_{\mathcal{F}_0}(\overline{Q})$ is spanned by an infinite set of commuting local functionals of u :

$$\begin{aligned} I_n &= \int dx \mathcal{J}_n(u(x)), \quad n = 1, 2, \dots \\ \{I_n, I_m\} &= 0 \end{aligned}$$

Evolutions of u with respect to the hamiltonians I_n form the quantum KdV hierarchy, while its evolution with respect to the hamiltonian $I_0 \equiv Q + \overline{Q}$ expressed in terms of field φ gives the sine-Gordon equation, which is a first equation in another integrable hierarchy [12].

3 Lattice Kac-Moody algebra and WZW model

3.1 Lattice Kac-Moody algebra – St.-Petersburg definition

In this paragraph we remind the definition of LKM due to Reshetikhin and Semenov-Tian-Shansky [26]. Physical application of this algebra appeared in the papers [27, 28], where the idea to consider the lattice regularization procedure was applied to the WZW model. Authors of [27, 28] proposed the following exchange relations for the quantum lattice L -operator (discrete analogue of the Kac-Moody current) was given

$$\begin{aligned} J(n)_1 J(n)_2 &= R^+ J(n)_2 J(n)_1 R^- \\ J(n+1)_1 R^- J(n)_2 &= J(n)_2 J(n+1)_1 \end{aligned} \quad (3.1)$$

We use the standard notation $A_1 \equiv A \otimes 1$, $A_2 \equiv 1 \otimes A$. R^+ and R^- are the two conjugated solutions

$$R^- = P(R^+)^{-1}P$$

of the Yang-Baxter equation (without spectral parameter)

$$R_{12}^\pm R_{13}^\pm R_{23}^\pm = R_{23}^\pm R_{13}^\pm R_{12}^\pm$$

For the sl_2 case these matrices have the following form

$$R^+ = q^{\frac{1}{2}} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1} - q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \quad R^- = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

For further purposes it will be more convenient for us to define another set of variables, analogous to Chevalley basis of sl_2 . Instead of matrix form of LKMA

$$J(n) = \begin{pmatrix} J(n)_{11} & J(n)_{12} \\ J(n)_{21} & J(n)_{22} \end{pmatrix}$$

with " sl_2 -constraint":

$$J(n)_{11} J(n)_{22} - q^{-1} J(n)_{21} J(n)_{12} = q^{\frac{1}{2}}$$

we choose coordinates

$$\begin{aligned} e_n &= J_n^{12} J_n^{22} \\ f_n &= J_n^{21} (J_n^{22})^{-1} \\ h_n &= (J_n^{22})^2 \end{aligned} \quad (3.2)$$

with exchange relations

$$\begin{aligned}
h_n h_{n+1} &= q h_{n+1} h_n \\
h_n e_n &= q e_n h_n & h_n f_n &= q^{-1} f_n h_n \\
h_n e_{n+1} &= q e_{n+1} h_n & h_n f_{n+1} &= q^{-1} f_{n+1} h_n \\
e_n f_n &= q^{-1} f_n e_n + q - 1 & [e_n, f_{n+1}] &= (q - 1) h_n
\end{aligned} \tag{3.3}$$

In quasiclassical limit (with the appropriate scaling of Poisson brackets) we obtain

$$\begin{aligned}
\{h_n, h_{n+1}\} &= h_n h_{n+1} \\
\{h_n, e_n\} &= h_n e_n & \{h_n, f_n\} &= (1) \\
\{h_n, e_{n+1}\} &= h_n e_{n+1} & \{h_n, f_{n+1}\} &= h_n f_{n+1} \\
\{e_n, f_n\} &= 1 + e_n f_n & \{e_{n+1}, f_n\} &= -h_n
\end{aligned} \tag{3.4}$$

3.2 Lattice WZW model

In some analogy with the continuous theory, one can formulate sort of WZW model on the lattice. Attempting to construct the lattice analogue of the WZW lagrangian

$$S[g] = \frac{k}{4\pi} \int tr(g^{-1} \partial_+ g)(g^{-1} \partial_- g) dx^+ \wedge dx_- + \frac{k}{12\pi} \int d^{-1} tr(g^{-1} dg)^{\wedge 3}$$

one meets some principal difficulties. Instead, in papers [27, 28, 29] it was proposed to take the classical equations of motion

$$\bar{\partial}(g^{-1} \partial g) = 0 = \partial(\bar{\partial} g g^{-1}) \tag{3.5}$$

and fundamental Poisson bracket, discussed in [30, 31, 32].

$$\{g(x)_1, g(y)_2\} = g(x)_1 g(y)_2 r^+ \theta(x - y) + r^- g(x)_1 g(y)_2 \theta(y - x)$$

as a starting point. Lattice analogue of eq. (3.5) is the following difference equation

$$g(n, \bar{n}) g(n+1, \bar{n})^{-1} = g(n, \bar{n}-1) g(n+1, \bar{n}-1)^{-1}$$

General solution of this equation has the form $g(n, \bar{n}) = g_L(n) g_R(\bar{n})$. Considering only one chiral sector, one can define the corresponding lattice Poisson structure for g_L (or g_R) as

$$\begin{aligned}
\{g(n)_1, g(m)_2\} &= g(n)_1 g(m)_2 r^\pm, \text{ when } \begin{pmatrix} n > m \\ n < m \end{pmatrix} \\
\{g(n)_1, g(n)_2\} &= r^+ g(n)_1 g(n)_2 + g(n)_1 g(n)_2 r^-
\end{aligned} \tag{3.6}$$

Quantum version of this bracket is given by the exchange relations

$$\begin{aligned}
g(n)_1 g(m)_2 &= g(m)_2 g(n)_1 R^\pm, \text{ when } \begin{pmatrix} n > m \\ n < m \end{pmatrix} \\
g(n)_1 g(n)_2 &= R^+ g(n)_2 g(n)_1 R^-
\end{aligned}$$

Lattice analogue of the continuous current $J^c = \frac{k}{2\pi} g \partial g^{-1}$ has the form

$$J(n) = g(n+1)g(n)^{-1}$$

and automatically obeys the exchange relations (3.1).

3.3 Lattice Wakimoto Construction

In this paragraph we describe the realization of LKMA in terms of free fields. Such a realization will be useful for the construction of the perturbed lattice WZW model. Recall first, that in continuous case there exists an explicit realization of the Wakimoto module over $\hat{\mathcal{G}}$ in terms of $r = \text{rank } \mathcal{G}$ free scalar fields

$$\mathcal{H}_a^i : \quad [j(k)^i, j(r)^j] = k\delta^{ij}\delta_{k+r} \quad [p^i, q^j] = -i\delta^{ij}$$

and $|\Delta_+|$ $\beta\gamma$ -systems

$$\mathcal{H}_{\beta\gamma}^\alpha : \quad [\beta(k)^\alpha, \gamma(r)^\alpha] = \delta_{k+r}$$

Such a realization was first obtained for $\widehat{sl(2)}_1$ by Wakimoto [33], then extended to arbitrary k by Zamolodchikov [34] and generalized for $\widehat{sl(n)}$ by Feigin and Frenkel [35]. Complete description of the complex of Wakimoto modules was given in the paper [36]. It was shown that the intertwining operators (screening charges) that build the complex can be obtained from contour integrals of the so-called screened vertex operators. Authors of [36] showed explicitly that SC realize the action of the $U_q(n_+)$ on the Fock space. Definition of screening operators appears already in the theory of realizations of finite dimensional Lie algebras in terms of differential operators. Generalizations of this definition to the case of affine Lie algebras are quite straightforward and explicit formulae for SC in terms of free fields can also be found in [36]. In the following we are going to build the lattice analogue of this construction. The space of local fields π_0 is defined as a space of finite-difference zero-degree polynomials of the following variables:

– lattice vertex operators a_n^i , corresponding to the simple roots α_i , with exchange relations

$$\begin{aligned} a_n^i a_{n+m}^j &= q^{A_{ij}/2} a_{n+m}^j a_n^i, \text{ for } m > 0, & A_{ij} - \text{Cartan matrix of } sl(n) \\ a_n^i a_n^{i+1} &= q^{1/2} a_n^{i+1} a_n^i \\ a_n^i a_n^j &= a_n^j a_n^i, \text{ when } |i - j| \geq 2 \end{aligned} \tag{3.7}$$

and Cartan-Weyl currents $p_n^i \equiv a_n^i (a_{n+1}^i)^{-1}$;

– lattice $\beta - \gamma$ systems, corresponding to the positive roots, with exchange relations

$$B_n^\alpha \Gamma_n^\alpha = q \Gamma_n^\alpha B_n^\alpha + q - 1 \equiv q \xi_n^\alpha - 1 \tag{3.8}$$

where we denoted $\xi_n^\alpha = 1 + \Gamma_n^\alpha B_n^{\alpha^{-1}}$. Values of the degree function are

$$\begin{aligned} \deg a_n^i &= \alpha_i \\ \deg B &= \deg \Gamma = 0 \end{aligned}$$

The space of local functionals is defined via the summation map

$$\sum : \pi_k \rightarrow \mathcal{F}_k$$

Using the realization of the screening charges in terms of the variables a_n^i , B_n^α , Γ_n^α , one needs to calculate the cohomology of the complex

$$\begin{aligned} \pi_0 &\xrightarrow{n_+(\mathcal{G})} \oplus_{i=1}^l \pi_{\alpha_i} \\ \deg \pi_{\alpha_i} &= \alpha_i \end{aligned}$$

Below we present explicit calculations for the sl_2 -case. Relations (3.7), (3.8) amount to:

$$\begin{aligned} a_n a_{n+m} &= q^{-1} a_{n+m} a_n \\ \beta_n \gamma_n &= q \gamma_n \beta_n + q - 1 \equiv q \xi_n - 1 \end{aligned} \tag{3.9}$$

Screening operators are given by the formulae:

$$\begin{aligned} Q_1 &\equiv Q_{\alpha_1} = \sum_n a_n \beta_n \\ Q_0 &\equiv Q_{\alpha_0} = \sum_n a_n^{-1} \gamma_n \end{aligned}$$

$\alpha_1 = (1, -1)$ is the simple root of $sl(2)$, $\alpha_0 = -\alpha_1$ is the affine root of $\widehat{sl(2)}$. Q_1 is the single generator of $U_{qn_+}(sl(2))$, and together with Q_0 they form Chevalley basis of $U_{qn_+}(\widehat{sl(2)})$. One finds by direct computation that the following combinations

$$\begin{aligned} e_n &= \beta_n \\ f_n &= \gamma_n - q^{-1/2} \gamma_{n-1} \xi_n p_{n-1} \\ h_n &= p_n \xi_n \xi_{n+1} \end{aligned} \tag{3.10}$$

obey the LKMA in Chevalley basis (3.4). In continuous limit formulae (3.10) coincide with the quasiclassical limit of Wakimoto construction [33, 34, 35]:

$$\begin{aligned} f(z) &= - : \gamma \gamma \beta : (z) - \sqrt{2(k+2)} \gamma(z) H(z) - k \partial \gamma(z) \\ h(z) &= 2 : \gamma \beta : (z) + \sqrt{2(k+2)} H(z) \\ e(z) &= \beta(z) \end{aligned}$$

¹We extensively use the exchange relations between ξ and original variables: $B\xi = q\xi B$, $\Gamma\xi = q^{-1}\xi\Gamma$.

It should be mentioned that for the first time Wakimoto bosonization on the lattice was proposed in the paper [29], where the authors using the twisted system of free fields, constructed the lattice analogue of Bernard-Felder cohomology. We give the general method of constructing the Wakimoto bosonization for the LKM over the untwisted system (3.7,3.8) of free fields, which seems to be simpler².

4 Lattice Drinfeld-Sokolov Reduction

4.1 DS reduction in algebraic formulation (sl_2 case)

The purpose of this paragraph is remind some essential facts about the continuous DS-reduction. We choose the algebraic formulation, which can be straightforwardly put on the lattice afterwards. In the rest of this paragraph we follow the paper [40]. Let us introduce a pair of ghost fields $b(z)$, $c(z)$ for the constraint $\chi(e(z)) = 1$. The corresponding BRST operator has the form $Q = Q_0 + Q_1$, where

$$Q_0 = \oint \frac{dz}{2\pi i} : c e : (z)$$

$$Q_1 = - \oint \frac{dz}{2\pi i} c(z) \chi(e(z))$$

Operators Q_0 , Q_1 satisfy the following relations

$$Q_0^2 = Q_1^2 = \{Q_0, Q_1\}_+ = 0$$

The corresponding W -algebra for sl_2 case is just a Virasoro algebra. The energy-momentum tensor has the form

$$T(z) = T^{sug}(z) + 2\partial h(z) + T^{gh}(z), \quad (4.1)$$

where $T^{sug}(z)$ is the Sugawara energy-momentum tensor

$$T^{sug}(z) = \frac{1}{2(h+2)} \left(: ef + fe : (z) + : h^2 : (z) \right)$$

and $T^{gh}(z) = : c\partial b : (z)$ is the ghost contribution.

It is well-known that the energy - momentum tensor (4.1) generates BRST- cohomology. Following [40] we will show that the BRST cohomology can be interpreted as the centralizer of screening charge. This will establish the connection with Fateev - Lukyanov construction of W -algebras in terms of free bosons [6, 7]. It should be mentioned that all the consideration can be naturally generalized for arbitrary $W\mathcal{G}$ -algebra. The realization of the BRST cohomology in terms of Fateev - Lukyanov approach is extremely useful for our further lattice consideration. In fact, B. Feigin's approach [41] to construction of lattice W -algebras is based on direct lattice setting of Fateev- Lukyanov construction. Lattice W -algebra is defined as a set of local lattice

²There is an "untwisting" transformation, relating the $\beta\gamma$ systems from Ref. [29] and ours, however even after substitution we get different realizations of the LKM.

fields generating the intersection of the kernels of appropriate system of screening charges (Feigin - Fuks operators).

The idea of reinterpreting of BRST-cohomologies in terms of screening charges construction is based on using the spectral sequence technique [44]. Let us apply this technique to the double complex with horizontal differential Q_0 and vertical differential Q_1 . In general, spectral sequence is an instrument of calculation $Q = Q_0 + Q_1$ -cohomologies in the framework of iterative procedure starting with Q_0 -cohomologies and improving them "perturbatively" on each step of certain procedure of construction a successive series of differential complexes converging to Q -cohomologies. In our case, however, spectral sequence collapses after the second correction, which gives the result

$$H_Q(*) \simeq H_{Q_1}(H_{Q_0}(*)).$$

It can be proved that Q_0 -cohomology is generated by the fields $\tilde{h}(z)$ and $c(z)$, where

$$\tilde{h}(z) = \alpha_+ h(z) + 2 : b c : (z)$$

However, the algebra of c and \tilde{h} in cohomologies is not free. The field $N(z) = \partial c(z) - : c \tilde{h} : (z)$ turns out to be Q_0 -exact. After factorization over this "null- field" $N(z)$ one can identify in Q_0 -cohomology

$$\begin{aligned} c &\longrightarrow \mathcal{V} = e^{-i\alpha_+ \phi(z)} \\ \tilde{h} &\longrightarrow H(z) = i\partial\phi(z) \end{aligned}$$

Of course, $c c$ OPE sector differs from $\mathcal{V}\mathcal{V}$ one, but this is inessential for the calculation of $H_{Q_1}(H_{Q_0}(*)).$ As there are no Q_0 -cohomologies at negative ghost numbers, one has

$$H_{Q_1}^{(0)} = \text{Ker} Q_1 = \text{Ker} \oint \frac{dz}{2\pi i} \mathcal{V}(z)$$

on the space of normally ordered differential polynomials of $H(z)$. The result is given by

$$T(z) = \frac{1}{2} : H^2 : (z) - (\alpha_+ + \alpha_-) \partial H(z).$$

This W_2 -algebra (= Virasoro in our case) comes naturally equipped with realization in terms of free bosonic field (without any relation of Wakimoto or some other bosonization of Kac-Moody currents).

4.2 Lattice Drinfeld-Sokolov Reduction

In this paragraph we generalize the previous procedure for the lattice case. We introduce lattice ghost system b_n, c_n with relations

$$\{b_n, c_m\} = \delta_{nm}$$

and lattice BRST-operator $Q = Q_0 + Q_1$, with

$$Q_0 = \sum_n c_n e_n, \quad Q_1 = \sum_n c_n.$$

Applying spectral sequence technique to the double complex, built on Q_1 and Q_0 we obtain quite analogously to continuum case that spectral sequence terminates on the second step

$$H_Q(*) \simeq H_{Q_1}(H_{Q_0}(*)).$$

The Q_0 -cohomologies are generated by lattice fields \tilde{h}_n and c_n , where

$$\tilde{h}_n = h_n(1 - b_n c_n)(1 - b_{n+1} c_{n+1}).$$

As its continuum counterpart the Poisson algebra for the fields \tilde{h}_n and c_n is not free in Q_0 -cohomologies for there exists lattice field $N_n = c_n \tilde{h}_n - c_{n+1}$, which is Q_0 -exact. The resulting Poisson algebra (factorized by the "null-vector" N_n) is isomorphic to the Poisson algebra for lattice "vertex operator" a_n and lattice $U(1)$ -current p_n (namely, it reproduces the latter algebra in $a - p$ and $p - p$ sectors after identification $c_n \leftrightarrow a_n$, $\tilde{h}_n \leftrightarrow p_n$). Thus calculation of Q_1 -cohomologies of complex $Q_0(*)$ reduces to Feigin's construction of the lattice W -algebra (lattice Virasoro or Faddeev-Takhtadjan-Volkov (FTV) algebra in our \hat{sl}_2 -case). Namely, we should find the kernel of screening operator

$$Q_1 = \sum_n a_n$$

This coincides with the cohomological definition of the lattice Virasoro generator [41].

Now we can calculate Q_1 -cohomological class in Q_0 -cohomologies, i.e. we should find an element constructed from c_n, \tilde{h}_n such that it is closed (and not exact) up to some null-field (lattice field containing $N_n = c_n \tilde{h}_n - c_{n+1}$ at least once). The answer is given in [41]. The following lattice field

$$\tilde{A}_n = \frac{1}{(1 + \tilde{h}_n)(1 + \tilde{h}_{n+1}^{-1})}$$

is an appropriate cohomology class

$$\{Q_1, \tilde{A}_n\} = -\hat{N}_n - \hat{N}_{n+1},$$

where \hat{N}_n is a null-field of the form

$$\hat{N}_n = \frac{N_n}{1 + \tilde{h}_n}.$$

It is easy to check that \tilde{A}_n forms FTV algebra

$$\begin{aligned} \{\tilde{A}_n, \tilde{A}_{n+2}\} &= \tilde{A}_n \tilde{A}_{n+1} \tilde{A}_{n+2} \\ \{\tilde{A}_n, \tilde{A}_{n+1}\} &= \tilde{A}_n \tilde{A}_{n+1} (-1 + \tilde{A}_n + \tilde{A}_{n+1}) \end{aligned} \tag{4.2}$$

It is also easy to verify that null-fields form an ideal in the Poisson algebra of fields constructed from c_n and \tilde{h}_n . This is a consequence of the following relations

$$\begin{aligned} \{c_{n+1}, N_n\} &= -c_n N_n \\ \{\tilde{h}_n, N_n\} &= \tilde{h}_n N_n \\ \{\tilde{h}_{n-1}, N_n\} &= \tilde{h}_{n-1} N_n \\ \{N_n, N_n\} &= -(c_{n-1} + c_n) N_n \end{aligned}$$

Notice that N_n is a "fermi-field", i.e. $N_n^2 = 0$.

One may be interested in construction of cohomological class B_n which is "better" than the found one \tilde{A}_n in such a way that $\{Q, B_n\} = 0$. In other words, B_n should represent the cohomology class of a double complex on the original phase space, not factorized over the null-field. In next section we will consider the Sugawara construction as an example of such a class. Here we explain how to organize the "improvement" process. The idea of construction of the class B_n is to find such corrections to $\ln \tilde{A}_n$ which kill \hat{N}_n terms. This can be done with the help of a staircase sequence in the double complex. Consider the following sequence

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow \{Q_0, \bullet\} & & & & \\
\ln \tilde{A}_n & \xrightarrow{\{Q_1, \bullet\}} & -(\tilde{N}_n + \tilde{N}_{n+1}) & & \\
& & \uparrow \{Q_0, \bullet\} & & \\
& & -(\phi_n + \phi_{n+1}) & \xrightarrow{\{Q_1, \bullet\}} & \phi_n \tilde{N}_n + \phi_{n+1} \tilde{N}_{n+1} \\
& & & & \uparrow \\
& & & & \frac{1}{2}\phi_n^2 + \frac{1}{2}\phi_{n+1}^2 \longrightarrow \dots
\end{array}$$

where $\phi_n = \frac{\tilde{f}_{n+1}}{1 + \tilde{h}_n}$. After the summation of this (infinite) staircase process one obtains

$$\begin{aligned}
\ln B_n &= \ln A_n + \sum_{m=1}^{\infty} \frac{\phi_n^m + \phi_{n+1}^m}{m} \quad \text{or} \\
B_n &= \frac{A_n}{(1 - \phi_n)(1 - \phi_{n+1})}
\end{aligned}$$

Through direct computation one verifies that B_n commutes with $Q = Q_0 + Q_1$ and obeys the same FTV - algebra (4.2) as \tilde{A}_n does.

5 Lattice Sugawara Construction

In this section we are going to discuss the analogue of the Sugawara construction on the lattice. The question of what object is to be considered an analogue of the Sugawara element is rather ambiguous, because it is not exactly clear what invariant property tells us that some element is the Sugawara-like one. Before we proceed with calculations, we make one comment, concerning the classical case. In continuum, the Sugawara element satisfies the second Gel'fand-Dickey Poisson algebra with *zero* central charge³. On the other hand the continuous limit of FTV algebra (4.2) reproduces the classical Virasoro algebra but with *non-zero* central term. This

³In quantum case, however, the central charge becomes non-zero due to quantum corrections.

makes one believe that the generator of the FTV algebra should contain some twisting part in the continuous limit independent on the underlying algebra it is built of. In the course of DS reduction such a twisted energy - momentum appears naturally and is given by

$$T(z) = \frac{1}{2(h+2)} : J^+ J^- + J^- J^+ + \frac{1}{2} J^0 J^0 : + \frac{1}{2} \partial J^0 + : \partial b c : . \quad (5.1)$$

Below we construct such a class A_n^{sug} which coincides in the continuous limit with (5.1).

For this purpose we start with a

$$A_n = \frac{1}{(1+h_n)(1+h_{n+1}^{-1})}$$

and after summation of a certain staircase process (slightly more complicated than the one constructed above) obtain the desired class

$$A_n^{\text{sug}} = \frac{1}{(1+h_n+\xi_n)(1+h_{n+1}^{-1}+\eta_{n+1})}, \quad (5.2)$$

where ξ_n and η_n are net corrections (after summation of a staircase process). The explicit form of ξ_n and η_n is

$$\begin{aligned} \xi_{2n} &= x_{2n}, & \xi_{2n+1} &= h_{2n+1} y_{2n+1} \\ \eta_{2n} &= y_{2n}, & \eta_{2n+1} &= h_{2n+1}^{-1} x_{2n+1} \end{aligned}$$

and

$$x_n = e_n f_{n+1} + (b_{n+1} - b_n) c_{n+1}, \quad y_n = e_{n+1} f_{n+1} h_n^{-1} + (b_{n+1} - b_n) c_n.$$

It is easy to see that the field (5.2) obeys FTV - algebra (4.2) and in the continuous limit (in the leading nontrivial order of a lattice spacing Δ) reduces to the classical limit of (5.1). After suppressing ghost fields ($b_n = c_n = 0$) one obtains twisted lattice Sugawara element. Naturally, $A_n^{\text{sug}}[b_n = c_n = 0]$ obeys the same FTV algebra.

In the end of this section, let us remind, that there exists another realization of the FTV algebra in terms of LKMA [46]. It has in somewhat the similar form

$$A_{2n} = \frac{1}{M_n^0} \quad A_{2n+1} = \frac{1}{M_n^1}$$

where

$$M_n^p = \frac{e_n f_{n+p}}{h_n h_{n+1} \dots h_{n+p-1}} \quad (5.3)$$

This realization turns out to be interesting in connection with lattice KP hierarchy, discussed in Section 7.2. Here we only would like to notice, that the fields M_n^p obey the nice *Lie* algebra of the w_∞ - type.

6 Perturbed lattice WZW model

6.1 Formulation of the Model

In this section we describe the construction, to which we refer to as "lattice perturbed WZW model" having in mind the parallelism with continuous case [45]. As in the Section 3 we will not construct any Lagrangian perturbation theory (in $1 + 1$ lattice space), but rather consider Hamiltonian perturbation in "one chiral sector" of the lattice WZW model.

We begin with construction of a lattice analogue of $\oint \Phi_{10}(z) \frac{dz}{2\pi i}$ perturbation operator where Φ_{jm} is a primary field of spin j with projection m . We choose

$$Q_0 = \sum_n a_n^{-1} \gamma_n.$$

for that purpose. To form nilpotent subalgebra $n_+ = e \oplus sl_2 \otimes \mathbf{C}[[t]]$ of affine algebra $\hat{sl}_2 = sl_2 \otimes \mathbf{C}(t)$ we add another screening

$$Q_1 = \sum_n a_n \beta_n$$

We introduce "deg" operator

$$\deg a_n = 1 \quad \deg a_n^{-1} = -1 \quad \deg \beta_n = \deg \gamma_n = 0$$

so that $\deg Q_0 = -1$, $\deg Q_1 = 1$ and the "correct" adjoint action as improved Poisson brackets

$$ad_A B := \{A, B\} - (\deg A \deg B) AB.$$

Then we have Serre relations

$$ad_{Q_0}^3 Q_1 = ad_{Q_1}^3 Q_0 = 0$$

fulfilled, so that the definition of perturbation operator is correct.

Let us consider now dynamical system with phase space of lattice fields of zero degree constructed from lattice $\beta - \gamma - a$ - system. Hamiltonian of the system is

$$H = Q_0 + Q_1. \tag{6.1}$$

The system corresponds to perturbed "chiral sector" of the lattice WZW model.

Our purpose now is to prove the integrability of this system and calculate the integrals of motion (IM) following the analogy with the continuous case, where the system with Hamiltonian (6.1) coincides with the Maxwell-Bloch eq. [45]. We will also give interpretation of the model in terms of lattice analogue of NLS hierarchy.

Let us start from an observation that the field h_n is a "zero mode" because it is conserved under the system evolution:

$$ad_H(h_n) = 0$$

This implies it is necessary to reduce our dynamical system and exclude the field h_n . We introduce new lattice fields

$$x_n = \beta_n a_n, \quad y_n = \gamma_n a_n^{-1}.$$

Corresponding Dirac brackets for these fields (up to a sign change) are

$$\begin{aligned}\{x_n, x_m\}_D &= -\text{sign}(n-m)x_n x_m, \\ \{y_n, y_m\}_D &= -\text{sign}(n-m)y_n y_m, \\ \{x_n, y_m\}_D &= \text{sign}(n-m)x_n y_m - \delta_{nm}(1+x_n y_m)\end{aligned}\tag{6.2}$$

In these variables Hamiltonian has the form

$$H = \sum_n (x_n + y_n).$$

Poisson algebra (6.2) strongly reminds that of from the Feigin- Enriquez model (FE) [47]. The only difference is the "central extension" (δ_{nm} term) in $x-y$ sector. This central term changes IM and dynamics drastically. Nevertheless, cohomological structure of the space of IM appears to be rigid with respect to such a deformation of Poisson brackets.

6.2 Interpretation of the model in terms of the NLS hierarchy

Before systematical study of IM we give a brief description of our model in terms of lattice analog of Nonlinear Schrodinger (NLS)- hierarchy. Renaming the variables $e_n \equiv \psi_n$, $f_n \equiv \bar{\psi}_n$ for better similarity we find that eqs. (6.2) reproduce exactly the first Poisson structure of the lattice analogue of NLS hierarchy.

$$\begin{aligned}\{\psi_n, \psi_m\} &= -\text{sign}(n-m)\psi_n \psi_m \\ \{\bar{\psi}_n, \bar{\psi}_m\} &= -\text{sign}(n-m)\bar{\psi}_n \bar{\psi}_m \\ \{\psi_n, \bar{\psi}_m\} &= -\text{sign}(n-m)\psi_n \bar{\psi}_m - (1 + \psi_n \bar{\psi}_n)\delta_{n,m} + \delta_{m,n+1}\end{aligned}$$

The pair of brackets (6.2) and $\{ , \}_0$ defined by

$$\{e_n, f_{n+1}\}_0 = 1$$

together with the two integrals

$$\begin{aligned}I_0 &= \sum_n \ln(1 + \psi_n \bar{\psi}_n) \\ I_1 &= \sum_n (\psi_n \bar{\psi}_{n+1})\end{aligned}$$

define a bihamiltonian system. Having bihamiltonian structure one can prove the existence of an infinite family of IM in involution. In next section we will give another proof of this fact based on consideration of deformed FE model. In continuum limit correspondence between few first IM's and their flows is as follows

$$\begin{aligned}I_0 &\rightarrow N \text{ (particle number),} \\ I_1 - I_0 &\rightarrow P \text{ (momentum),} \\ I_2 - 2I_1 + I_0 &\rightarrow \text{(NLS Hamiltonian),}\end{aligned}$$

where

$$I_2 = \sum_n \left(\frac{\psi_n^2 \bar{\psi}_{n+1}^2}{2} - \psi_n \bar{\psi}_{n+2} \right)$$

6.3 Hidden FTV algebra

There is an interesting hidden symmetry on the reduced phase space. Namely, the variables $N_n^0 = e_n f_n = \psi_n \bar{\psi}_n$ and $N_n^1 = e_n f_{n+1} = \psi_n \bar{\psi}_{n+1}$ obey FTV algebra (4.2) under the following identification

$$A_{2n} = \frac{1}{N_n^0}, \quad A_{2n+1} = \frac{1}{N_n^1}. \quad (6.3)$$

Let us mention about strange “notational” similarity here. If e_n and f_n were the fields from the Chevalley basis of lattice sl_2 , one would have the same formulae (6.3) for the generators of the FTV algebra on the phase space reduced under $h_n = 1$ constraint [46].

6.4 Deformed FE model

In this section we will give an alternative proof of the existence of an infinite system of IM for our model with Hamiltonian (6.1). Consider the following one-parameter deformation of FE model

$$\begin{aligned} \{x_n, x_m\}_D &= -\text{sign}(n-m)x_n x_m \\ \{y_n, y_m\}_D &= -\text{sign}(n-m)y_n y_m \\ \{x_n, y_m\}_D &= \text{sign}(n-m)x_n y_m - \delta_{nm}(\lambda + x_n y_m) \end{aligned} \quad (6.4)$$

with Hamiltonian

$$H = Q_+ + Q_-,$$

where $Q_+ = \sum_n x_n$ and $Q_- = \sum_n y_n$. For the lattice variables x_n and y_n we have

$$\deg x_n = 1, \quad \deg y_n = -1.$$

For $\lambda = 0$ we obtain FE model and for $\lambda = 1$ we come to our initial algebra (6.2) corresponding to perturbed lattice WZW (or lattice NLS) model. It should be mentioned that all the Poisson algebras \mathcal{A}_λ defined by bracket (6.4) are pairwise isomorphic for $\lambda \in (0, \infty)$.

In the paper [47] IM for the system (6.4) for $\lambda = 0$ have been expressed in terms of cohomology classes. Standard arguments give that the ring of cohomologies does not change under the infinitesimal variation of basic algebraic structure (6.4) on the phase space. The isomorphism of algebras $\mathcal{A}_{\lambda \neq 0}$ allows us to replace an infinitesimal deformation by the finite one. Thus, the rings of cohomologies for FE model and perturbed lattice WZW model are the same.

7 Embedding of the lattice NLS hierarchy into the lattice KP hierarchy

7.1 Nonlinear Lattice W_∞ algebra .

We briefly remind how Feigin's construction of LW_N - algebra can be extended to the case of $N = \infty$ [39]. We restrict ourselves with quasiclassical case. Let us introduce the following set of lattice variables $\{a_n^j\}_{j=1}^N$ with the following Poisson structure

$$\begin{aligned}\{a_n^i, a_m^i\} &= \text{sign}(m - n) a_n^i a_m^i \\ \{a_n^i, a_n^{i+1}\} &= -\frac{1}{2} a_n^i a_n^{i+1} \\ \{a_n^i, a_m^{i+1}\} &= -\frac{1}{2} \text{sign}(m - n) a_n^i a_m^{i+1}\end{aligned}\tag{7.1}$$

Values of the degree function on these fields are as usual

$$\deg(a_n^i) = 1, \quad \deg((a_n^i)^{-1}) = -1 \quad i = 1, \dots, N - 1$$

Denote by Π_n the space of finite difference polynomials of degree n . Following the papers [41, 38, 39, 25] we define lattice W_N algebra (LW_N) as an intersection

$$\cap_{i=1}^{N-1} \text{Ker}(\text{ad}_{Q_i}) \cap \Pi_0\tag{7.2}$$

where $Q_i = \sum_n a_n^i$ are the corresponding *screening charges*. This intersection is known to be spanned by $N - 1$ generators $L_n \equiv W_n^{(2)}, W_n^{(3)}, \dots, W_n^{(N)}$. In the limit $N \rightarrow \infty$ we obtain lattice analogue of classical non-linear W_∞ -algebra. Now let us complete this program explicitly. To construct the generators $W_n^{(i)}$ we will use more convenient variables than a_n^i . First of all, we exclude no-zero degree components of the phase space by using as the basic variables the lattice analogues of the Cartan currents of sl_N , associated with simple roots $\alpha_1, \alpha_2, \dots$:

$$p_n^i = a_n^i (a_{n+1}^i)^{-1}$$

Calculations with these variables turned out to be rather tedious [38], so this time we choose another basis in the root system of sl_N and use Weyl chamber generators $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots$:

$$\begin{aligned}k_n^1 &= p_n^1 \\ k_n^2 &= p_n^1 p_n^2 \\ k_n^3 &= p_n^1 p_n^2 p_n^3 \\ &\dots \dots \dots, \\ k_n^{N-1} &= p_n^1 p_n^2 \dots p_n^{N-1}\end{aligned}\tag{7.3}$$

The following combinations turn out to be the best for our purposes:

$$\begin{aligned}\alpha_n^1 &= \sum_{i=1}^{N-1} k_n^i \\ \alpha_n^2 &= \sum_{i=1}^{N-1} \sum_{j=i+1}^N k_n^i k_{n+1}^j \\ &\dots \dots \dots \\ \alpha_n^p &= \sum_{\{u(l)\}} \prod_{j=1}^{p-1} k_{n+j}^{u(j)}\end{aligned}$$

where the summation goes over all sets $\{u(j)\}$ such that $u(j+1) > u(j)$. They form a quadratic Poisson algebra

$$\{\alpha_n^p, \alpha_{n+m}^q\}_1 = \theta_m^{pq} (-\alpha_n^p \alpha_{n+m}^q + \alpha_n^{q+m} \alpha_{n+m}^{p-q}), \quad (7.4)$$

where $\theta_m^{pq} = \theta(p-m)\theta(q-p+m-1)$ and $\theta(x)$ is a step function $\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$. The index 1 of the bracket indicates that this is the analogue of *first* Gelfand-Dickey (GD) structure for the N -KdV hierarchy. Generators of LW_N form the analogue of the *second* GD structure. It turns out that quite parallel to the continuous case there are Miura transformation, relating the α - fields (7.4) with the generators of LW_N . Direct calculation (the most convincing method of proof) shows that the following fields commute with all screening operators ($i = 2, 3, \dots, N$):

$$\begin{aligned}W_n^{(i)} &= \frac{\alpha_{n+1}^{i-1} + \alpha_n^i}{(1 + \alpha_n^1) \dots (1 + \alpha_{n+i-1}^1)}, \quad i = 2, 3, \dots, N-1 \\ W^{(N)} &= \frac{\alpha_{n+1}^{N-1}}{(1 + \alpha_n^1) \dots (1 + \alpha_{n+N-1}^1)}\end{aligned} \quad (7.5)$$

In the limit $N \rightarrow \infty$ one find the brackets between the fields $W^{(i)}$. Putting $W^{(1)} \equiv 1$, we have

$$\begin{aligned}\{W_n^p, W_{n+m}^q\} &= W_n^p W_{n+m}^q (1 - W_{n+m-1}^q - W_{n+p}^q) - W_n^{q+m} W_{n+m}^{p-m} \\ &\quad - W_n^{p+1} W_{n+m}^q + W_n^p W_{n+m-1}^{q+1}, \quad \text{for } m \leq p, \quad q+m \geq p+1, \\ \{W_n^p, W_{n+m}^q\} &= W_n^p W_{n+m}^q (-W_{n+m-1}^q + W_{n+m+p}^q) \\ &\quad - W_n^p W_{n+m-1}^{q+1} - W_n^p W_{n+m}^{q+1}, \quad \text{for } m \geq 1, \quad p \geq m+q+1. \\ \{W_n^p, W_{n+p+1}^q\} &= -W_n^p W_{n+p}^1 W_{n+p+1}^q - W_n^{p+q+1} + W_n^{p+1} W_{n+p+1}^q + W_n^p W_{n+p}^{q+1}, \\ \{W_n^p, W_{n+p-q}^q\} &= -W_n^p W_{n+p-q-1}^1 W_{n+p-q}^q W_n^p W_{n+p-q-1}^{q+1}, \quad \text{for } p \geq q+1, \\ \{W_n^p, W_n^q\} &= -W_n^p W_{n+p}^1 W_n^q W_n^{p+1} W_n^q, \quad \text{for } q \geq p+1,\end{aligned} \quad (7.6)$$

Notice that algebra (7.6) is *unhomogeneous* with respect to natural gradation: if we ascribe to the lattice field $\deg W_n^p = p$, we find that the r.h.s. of (7.6) has degree greater or equal than l.h.s. It is possible, however, to split the bracket (7.6) in two parts $\{, \} = -\{, \}_1 + \{, \}_2$, where $\{, \}_1$ is defined by (7.4) and $\deg \{, \}_1 = 0$, $\deg \{, \}_2 = 1$. Ending this section, we would like to highlight several points:

- Distinct from continuous case, for any *finite* N , algebra LW_N does not form a subalgebra of LW_∞ . However, by forcing $W^{(i)} = 0$ for $i \geq N$ one can obtain any LW_N .
- In continuous case there exists the so-called em two-boson realization of KP hierarchy [48], in which W_∞ -algebra generators are expressed in terms of two $u(1)$ currents. Analogous construction happens to exist on the lattice. Fields forming Poisson algebra (7.6) can be realized in terms of two lattice $u(1)$ currents [39] $u_n = t_{2n}$ and $v_n = t_{2n+1}$, commuting as

$$\{t_n, t_{n+1}\} = -t_n t_{n+1} \quad (7.7)$$

- Under properly defined continuous limit the brackets 1 and 2 become the corresponding Poisson structures of the KP-hierarchy (resp. linear w_∞ and non-linear W_∞ algebras).

7.2 Integrable model associated with $LWA_{+\infty}$ algebra

Define the affine vertex of \widehat{sl}_N as $a_n^0 = \prod_{i=1}^{N-1} (a_n^i)^{-1}$. The corresponding screening operator associated with the imaginary root of sl_N is

$$Q_0 = \sum_{n \in \mathbf{Z}} a_n^0$$

Differential $\hat{Q} = Q_0 + Q = \sum_{j=0}^N Q_j$ may be considered as the hamiltonian of \widehat{sl}_N - Toda system. According to definitions of the work [3], Space of Integrals of Motion of this system is defined as an intersection

$$Ker(ad_{Q_0}) \cap Ker(ad_{Q_1}) \cap \dots \cap Ker(ad_{Q_{N-1}}) \cap \frac{\Pi_0}{\partial \Pi_0} \quad (7.8)$$

The word *integrals* is encoded in the last intersection because of obvious isomorphism

$$\frac{\Pi_0}{\partial \Pi_0} \cong \Pi_0^{int} \leftarrow \Pi_0 : \sum_n$$

Before describing the space (7.8), let us take look at a simpler problem. It is almost a trivial statement, that a system associated with the pair of brackets $\{, \}_1$ and $\{, \}_2$ is integrable, with infinite number of conservation laws in involution. One just have to have *two* integrals, commuting under both brackets. The simplest choice is [38]

$$I^{(1)} = \sum_n W_n^{(2)}$$

$$I^{(2)} = \sum \left(\frac{(W_n^{(2)})^2}{2} + W_n^{(2)} W_{n+1}^{(2)} - W_n^{(3)} \right)$$

The subsequent procedure is obvious: using the bi-hamiltonian structure, one can easily obtain the whole series of conservation laws in involution by the recursive procedure. The answer for

any N (essentially, including $N = \infty$) can be found in [39]. We rewrite it here for completeness. For given N , the series is given by

$$I_N^{(k)} = \frac{1}{k} \text{Tr} \left(\mathcal{L}_N^k \right) \quad (7.9)$$

where Lax matrix L_N is defined through its inverse

$$(\mathcal{L}_N)_{n,m}^{-1} = \delta_{n,m+1} - W_n^{(2)} \delta_{n,m} + W_n^{(3)} \delta_{n,m-1} - \dots + (-1)^{N-1} W_n^{(N)} \delta_{n,m-N+1} \quad (7.10)$$

Introduce the translation matrix $\Lambda_{n,m} = \delta_{n,m-1}$ and diagonal matrices $W_{n,m}^{(i)} = W_n^{(i)} \delta_{n,m}$. In the compact notations L -operator (7.10) has the form

$$\mathcal{L}_N = \Lambda \cdot \frac{1}{1 - L \cdot \Lambda + W^{(3)} \cdot \Lambda^2 - \dots + (-1)^{N-1} W^{(N)} \Lambda^{N-1}}. \quad (7.11)$$

It really is the L -operator of our dynamical system, because the evolution equations can be written in the form

$$\frac{\partial \mathcal{L}}{\partial t_p} = [\mathbf{A}_p, \mathcal{L}], \quad (7.12)$$

where $\mathbf{A}_N^{(p)} = (\mathcal{L}_N^p)_+$. Now let us return to our original problem of description of the space (7.8). Clearly, all the integrals described above commute with $Q = \sum_{j=0}^N Q_j$ even *locally* by the construction (7.2). In addition, direct calculation shows, that *they also commute with Q_0* . The last step to make is to notice, that of N vertex operators corresponding to \widehat{sl}_N we needed only $N-1$ corresponding to simple roots to construct LWA_N generators and integrals of motion. In principle, we could pick up *any* $N-1$ vertex operators, and follow the same steps. Eventually, the space of integrals of motion for \widehat{sl}_N Toda system can be described in terms of generating functions as

$$R_{\widehat{SL}_N}(\lambda_1, \dots, \lambda_N) = \sum_{i=1}^N R_{SL_N}^{(i)}(\lambda_i) \quad (7.13)$$

where $R_{SL_N}^{(i)}(\lambda_i) = \sum_{s=0}^{\infty} I^{(s)} \lambda_i^s$ is the generating function for the conservation laws of the lattice N -KdV hierarchy, associated with the roots $\{\#1, \#2, \dots, \#i-1, \#i+1, \dots, \#N\}$.

Finally, it is quite obvious that all the formulae and constructions above apply directly to the case of $N = \infty$.

$$M_j = \sum_{l=1}^j m_l, \quad M_0 = 0.$$

7.3 Embedding of the Lattice NLS into the Lattice KP hierarchy

It can be proved that evolution of the previously defined lattice fields

$$M_n^p = \frac{e_n f_{n+p}}{h_n h_{n+1} \dots h_{n+p-1}}$$

is consistent with lattice KP hierarchy (7.12) under the following identification.

$$L_{n,n+p} = (-1)^p M_n^{p+1}.$$

Recall, however, that as defined by eq. (5.3), variables $\{M_n^p\}_{p=1}^\infty$ are not functionally independent. There is a set of quadratic relations, like $M_n^2 M_{n+1}^2 = M_n^3 M_{n+1}^1$, which may be interpreted as Plucker relations of some Grassmanian. Two independent generators of the whole family are M_n^0 and M_n^1 , which form FTV algebra. Thus, one may understand the lattice NLS embedding into the lattice KP as a two *non-abelian* field realization of lattice KP. To compare this with the *abelian* two-field realization, mentioned in the end of Section 7.1, we notice that FTV algebra (4.2) and lattice $u(1)$ algebra (7.7), when treated just as Poisson structures, can be viewed as respectively *second* and *first* brackets for the Volterra hierarchy [17, 21].

8 Concluding remarks

In this paper we have studied the lattice analogs of various conformal theories as well as their integrable perturbations. We have found that when described in proper invariant terms, many of the well-known continuous constructions have their match on the lattice. Besides, reiterating the ideas from the Introduction, we have discovered, that on the lattice analogues of conformal theories and integrable models admit description in universal terms. We explicitly described for the first time lattice analogues of the Drinfel'd-Sokolov reduction and of the Sugawara construction. In the framework of the lattice WZW the lattice Sugawara energy-momentum tensor was constructed. Then we described lattice MB system as a “chiral perturbation” of the lattice WZW model by the field of spin one. Evolution equations under the integrals of motion of this system form the integrable lattice NLS hierarchy. Finally, we found an embedding of the lattice NLS hierarchy to the lattice KP hierarchy, which seems to be very natural in continuous case.

There is still an open question how to give the geometrical description of the lattice MB system using the Lie group cosets, in analogy with continuous case [45].

We described the Spaces of IM's for several integrable systems discussed in the paper, using Lax representation and bi-hamiltonian structure. It would be extremely interesting to compare the results of cohomological [47] and St.-Petersburg approaches [42, 18] with our answers. Recently S. Kryukov calculated first three integrals in the quasiclassical limit of lattice sine-Gordon theory [49], using the generating function from the paper [42]. After careful comparison, we found that his integrals of motion can be expressed in terms of certain linear combinations of our ones.

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References

- [1] B. Feigin, E. Frenkel, Phys. Lett. **B276** (1992), 79

- [2] B. Feigin, E. Frenkel, *Integrals of Motion and Quantum Groups* (Nov. 1993), Yukawa Inst. preprint **YITP/K-1036**.
- [3] B. Feigin, E. Frenkel, *Kac-Moody Groups and Integrability of Soliton Equations* (Nov. 1993, revised Nov. 1994).
- [4] V.I. Dotsenko, V. Fateev, Nucl. Phys. **B240**[FS12] (1984), 312; Nucl. Phys. **B251**[FS13] (1985), 691.
- [5] G. Felder, Nucl. Phys. **B317** (1989), 215; erratum, Nucl. Phys. **B324** (1989), 548.
- [6] V. Fateev, S. Lukyanov, Sov. Phys. JETP **67** (3) (1988), 447.
- [7] V. Fateev, S. Lukyanov, Int. J. Mod. Phys. **A3** (1988), 507.
- [8] C. Gomés, G. Sierra, *Quantum Group Symmetry of Rational Conformal Field Theories*, Geneva Univ. preprint **UGVA-DPT-90/04-669** (1990).
- [9] A. Gerasimov, *Quantum Group Structure in Minimal Models*, Moscow preprint **ITEP-91-22** (1991).
- [10] R. Sasaki, I. Yamanaka, Adv. Studies in Pure Math. **16** (1988), 271.
- [11] M. Freeman, P. West, *On the relation between integrability and infinite-dimensional algebras*, in *Pathways to Fundamental Theories*, eds. L. Brink, R. Marnelius, World Scientific, Singapore (1993), p. 21.
- [12] L. Faddeev, L. Takhtadjan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin (1987).
- [13] V. Korepin, G. Izergin, N. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz*, Cambridge Univ. Press (1992)
- [14] L. Faddeev, *Algebraic Aspects of Bethe Ansatz*, Lectures at Stony-Brook **ITP-SB-94-11** (March, 1994).
- [15] L. Dickey, *Soliton Equations and Hamiltonian Systems*, Adv. Ser. in Math. Phys. **12**, World Scientific, Singapore (1990)
- [16] *New Symmetry Principles in Quantum Field Theory*, eds. J. Fröhlich et al., Plenum, New York (1992).
- [17] L. Faddeev, L. Takhtadjan, Lect. Notes in Phys. **246** (1986), 66.
- [18] E. Frenkel, *W-algebras and Langlands-Drinfeld correspondence*, in Ref. [16], p. 433.
- [19] O. Babelon, Phys. Lett. **B238** (1990), 234.
- [20] O. Babelon, L. Bonora, Phys. Lett. **B253** (1991), 365.
- [21] A. Volkov, Theor. Math. Phys. **74** (1988), 135 (in Russian).

- [22] A. Volkov, Phys. Lett. **A167** (1992), 345.
- [23] E. Witten, Comm. Math. Phys. **121** (1989), 351.
- [24] G. Moore, N. Seiberg, Phys. Lett. **B220** (1989), 422.
- [25] Ya. Pugai, Theor. Math. Phys. **100** (1994), 132.
- [26] N. Reshetikhin, M. Semenov-Tian-Shansky, Lett. Mat. Phys. **19** (1990), 133.
- [27] A. Alekseev, L. Faddeev, M. Semenov-Tian-Shansky, *The unravelling of Quantum Group Symmetry in the WZNW model* (Jan. 1991), preprint **CERN-5981/91**.
- [28] A. Alekseev, L. Faddeev, M. Semenov-Tian-Shansky, *Hidden Quantum Groups inside Kac-Moody algebra*, Lec. Not. Math. **1510** (1992), 148.
- [29] F. Falceto, K. Gawedzki, J. Geom. Phys. **11** (1993), 251.
- [30] A. Alekseev, S. Shatashvili, Comm. Math. Phys. **133** (1990), 353.
- [31] B. Blok, Phys. Lett. **B233** (1989), 359.
- [32] L. Faddeev, Comm. Math. Phys. **132** (1992), 131.
- [33] M. Wakimoto, Comm. Math. Phys. **104** (1986), 605.
- [34] A. Zamolodchikov, unpublished.
- [35] B. Feigin, E. Frenkel, *Affine Kac-Moody algebras and bosonization*, in *Physics and Mathematics of Strings*, V. G. Knizhnik Memorial Volume, eds. L. Brink et al., World Scientific, Singapore (1990), p. 271.
- [36] P. Bouwknegt, J. McCarthy, K. Pilch, Comm. Math. Phys. **131** (1990), 125.
- [37] A. Goddard, K. Olive, Int. J. Mod. Phys. **A1** (1986), 303.
- [38] A. Belov, K. Chaltikian, Phys. Lett. **B309** (1993), 268.
- [39] A. Belov, K. Chaltikian, Phys. Lett. **B317** (1993), 64.
- [40] P. Bouwknegt, K. Schoutens, *W-symmetry in CFT*, Phys. Repts. **223** (1993), 183.
- [41] B. Feigin, Talk at St.-Petersburg conference on Geometry and Physics (Sept. 1992).
- [42] L. Faddeev, A. Volkov, Phys. Lett. **B315** (1993), 311.
- [43] L. Faddeev, *Current-like variables in massive and massless integrable models*, preprint **hep-th/9408041**
- [44] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York (1982).

- [45] A. Antonov, A. A. Belov, B. Feigin, *Geometrical Description of the Local Integrals of Motion of the Maxwell-Bloch equation*, preprint **LANDAU-94-TMP-8**, **hep-th/9501128**.
- [46] A. Belov, K. Chaltikian, Phys. Lett. **B317** (1993), 73.
- [47] B. Feigin, B. Enriquez, *Integrals of Motion of Classical lattice sin-Gordon system*, **hep-th/9409075**.
- [48] F. Yu, Y.-S. Wu, Phys. Rev. Lett. **68** (1992), 2996.
- [49] S. Kryukov, Landau Inst. preprint March, 1995 (in Russian).